EXERCISE – V HINTS & SOLUTIONS

Sol.1 **(a)** A

x_1, x_2, x_3 and y_1, y_2, y_3 are is G.P.

$x_2 = x_1 r$; $x_3 = x_1 r^2$

$y_1 = y_1 r$; $y_3 = y_1 r^2$

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_1 r & y_1 r & 1 \\ x_1 r^2 & y_1 r^2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}x_1 y_1 \begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix} = 0$$

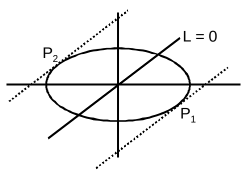
So Three points lie on line.

(b) $4x^2 + ay^2 = 1$

$L : 8x - ay = 0$

As tangents are parallel to

$L = 0$



So L = 0 will touch the Ellipse in 2nd and 4th quadrant. So Ans. (B) (D) by Symmetry.

(c) Eqn of any tangent to circle $x^2 + y^2 = r^2$ is

$x \cos\theta + y \sin\theta = r$ .....(1)

If Eqn (1) is tangent to $4x^2 + 25y^2 = 100$

or $\frac{x^2}{25} + \frac{y^2}{4} = 1$ at (x_1, y_1)

$$\frac{xx_1}{25} + \frac{yy_1}{4} = 1 \qquad .....(2)$$

Eqn (1) & (2) are same

$$\frac{x_1/25}{\cos\theta} = \frac{y_1/4}{\sin\theta} = \frac{1}{r}$$

$$x_1 = \frac{25\cos\theta}{r}; \; y_1 = \frac{4\sin\theta}{r}$$

The line (1) meet the coordinates axes is A(r sec θ, 0) and B(0, r cosec θ), Let (h, k) be mid point of AB

Then $h = \frac{r\sec\theta}{2}$ and $k = \frac{r\,\text{cosec}\,\theta}{2}$

$2h = \frac{r}{\cos\theta}$ and $2k = \frac{r}{\sin\theta}$

$x_1 = \frac{25}{2h}$ and $y_1 = \frac{4}{2k}$

As (x_1, y_1) lis on the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$

$$\frac{1}{25}\left(\frac{625}{h^2}\right) + \frac{1}{4}\left(\frac{4}{k^2}\right) = 1$$

$$\frac{25}{4h^2} + \frac{1}{k^2} = 1$$

$25k^2 + h^2 = 4h^2 k^2$

Locus : $25y^2 + x^2 = 4x^2 y^2$

Sol.2 Ellipse : $\frac{x^2}{16} + \frac{y^2}{4} = 1$; $a^2 = 16$, $b^2 = 4$

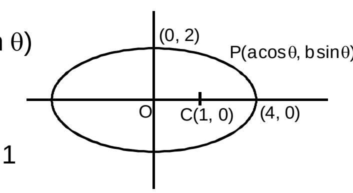
Let the Equation of circle

$(x - 1)^2 + y^2 = r^2$

Bcoz circle is biggest that means it will touch the ellipse at

P (a cosθ, b sin θ)

Tangent at P



$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Slope of Tangent $m_1 = -\frac{b}{a}\,\frac{\cos\theta}{\sin\theta}$

$$= -\frac{1}{2}\,\frac{\cos\theta}{\sin\theta}$$

m_2 = slope of CP

$$= \frac{b\sin\theta}{a\cos\theta - 1} = \frac{2\sin\theta}{4\cos\theta - 1}$$

And $m_1 m_2 = -1$

$$-\frac{1}{2}\,\frac{\cos\theta}{\sin\theta} \times \left(\frac{2\sin\theta}{4\cos\theta - 1}\right) = -1 \Rightarrow \cos\theta = \frac{1}{3}$$

Radius of cicle = $\sqrt{(a\cos\theta - 1)^2 + b^2 \sin^2\theta}$

$$= \sqrt{\left(\frac{4}{3} - 1\right)^2 + 4\left(1 - \frac{1}{9}\right)^2}$$

$r = \sqrt{\frac{11}{3}}$

Equation of circle will be $(x-1)^2 + y^2 = \frac{11}{3}$

Sol.3 Let the co-ordinate of A $\equiv$ (a cosθ, a sinθ) so that the coordinates of B $\equiv$

$$\left\{a\cos\left(\theta+\frac{2\pi}{3}\right), a\sin\left(\theta+\frac{2\pi}{3}\right)\right\}$$

$$C \equiv \left\{a\cos\left(\theta+\frac{4\pi}{3}\right), a\sin\left(\theta+\frac{4\pi}{3}\right)\right\}$$

$\Rightarrow$ P (a cosθ, b sinθ)

$$Q\ \left\{a\cos\left(\theta+\frac{2\pi}{3}\right), b\sin\left(\theta+\frac{2\pi}{3}\right)\right\}$$

$$R\ \left\{a\cos\left(\theta+\frac{4\pi}{3}\right), b\sin\left(\theta+\frac{2\pi}{3}\right)\right\}$$

[It is given that P,Q,R are on the same side of x – axis as A,B and C]
equation of normal to the ellipse at P is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

or ax sinθ – by cosθ = $\frac{1}{2}$ $(a^2 - b^2)$ sin2θ ....(1)

equation of normal to the ellipse at Q is

$$ax\sin\left(\theta+\frac{2\pi}{3}\right) - by\cos\left(\theta+\frac{2\pi}{3}\right)$$

$$= \frac{1}{2}(a^2-b^2)\sin\left(2\theta+\frac{4\pi}{3}\right) \quad ....(2)$$

equation of normal to the ellipse at R is

$$ax\sin\left(\theta+\frac{4\pi}{3}\right) - by\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$= \frac{1}{2}(a^2-b^2)\sin\left(2\theta+\frac{8\pi}{3}\right) \quad ....(2)$$

But $\sin\left(\theta+\frac{4\pi}{3}\right) = \sin\left(2\pi+\theta-\frac{2\pi}{3}\right) = \sin\left(\theta-\frac{2\pi}{3}\right)$

$$\cos\left(\theta+\frac{4\pi}{3}\right) = \cos\left(2\pi+\theta-\frac{2\pi}{3}\right) = \cos\left(\theta-\frac{2\pi}{3}\right)$$

$$\sin\left(2\theta+\frac{8\pi}{3}\right) = \sin\left(4\pi+2\theta-\frac{4\pi}{3}\right) = \sin\left(2\theta-\frac{4\pi}{3}\right)$$

Now equation (3) can be written by

$$ax\sin\left(\theta-\frac{2\pi}{3}\right) - by\cos\left(\theta-\frac{2\pi}{3}\right)$$

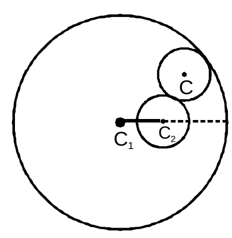
$$= \frac{1}{2}(a^2-b^2)\sin\left(2\theta-\frac{4\pi}{3}\right) \quad .....(4)$$

For the lives (1), (2) and (4) are concerrent

$$\text{Area} = \begin{vmatrix} a\sin\theta & -b\cos\theta & \frac{1}{2}(a^2-b^2)\sin 2\theta \\ a\sin\left(\theta+\frac{2\pi}{3}\right) & -b\cos\left(\theta+\frac{2\pi}{3}\right) & \frac{1}{2}(a^2-b^2)\sin\left(2\theta+\frac{4\pi}{3}\right) \\ a\sin\left(\theta-\frac{2\pi}{3}\right) & -b\cos\left(\theta-\frac{2\pi}{3}\right) & \frac{1}{2}(a^2-b^2)\sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

= 0 $\Rightarrow$ points are concurrent.

Sol.4 Let centres of given circles be c_1 & c_2 and that of moving circle be c
And radius R_1 & R_2 and R
$|C_1C_2| < R_1 - R_2$ ...(1) $R_1 > R_2$
$|C_2C| = R_2 + R$ ...(2)
$|C_1C| = R_1 - R$ ...(3) $R_2 > R$
(2) + (3)
$|C_2C| + |C_1C| = R_1 + R_2$ = constant
That means that locus of centre of moving circle be an ellipse whose focus be given by C_1 & C_2 & length of major axis be $R_1 + R_2$.
If C_1 & C_2 same than it is a circle cocentric with the



given circles & having radius $\frac{R_1+R_2}{2}$.

Sol.5 px + qy = r ....(1)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad |\theta - \phi| = \frac{\pi}{4}$$

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right) \quad ...(2)$$

(1) and (2) are same

$$\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{ap}=\frac{\sin\left(\theta+\frac{\phi}{2}\right)}{bq}=\frac{\cos\left(\frac{\theta-\phi}{2}\right)}{r}=\frac{\cos\frac{\pi}{8}}{r}$$

$$\cos\left(\frac{\theta+\phi}{2}\right)=\frac{pa}{r}\cos\frac{\pi}{8}$$

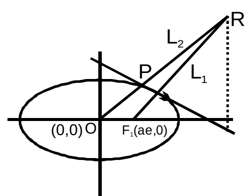
$$\sin\left(\frac{\theta+\phi}{2}\right)=\frac{bq}{r}\cos\frac{\pi}{8}$$

Square & add

$$1=\left[\left(\frac{Pa}{r}\right)^2+\left(\frac{bq}{r}\right)^2\right]\cos^2\frac{\pi}{8}$$

$$a^2p^2+b^2q^2=r^2\sec^2\frac{\pi}{8}$$

Sol.6 Ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



P(a cosθ, b sin θ)
Tangent at P

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \quad ...(1)$$

Equation of line OP ($L_2 = 0$) : $y=\frac{b}{a}\tan\theta\, x$ ...(2)

Slope of line L_1 which is $\perp^n$ to tangent & passing throug focus

$$m_2=-\frac{1}{m}=\frac{a\tan\theta}{b}$$

So equation of line L_1

$$y-0=\frac{a\tan\theta}{b}(x-ae)$$

$$y=\frac{a\tan\theta}{b}(x-ae) \quad ...(3)$$

Solving (2) and (3)

$$\frac{b}{a}\tan\theta\,.\,x=\frac{a}{b}\tan\theta\,(x-ae)$$

$$\frac{b^2-a^2}{ab}\,x=-\frac{a^2e}{b}$$

$$\frac{a^2-b^2}{a^2}\,x=ae \qquad \frac{a^2-b^2}{a^2}=e^2$$

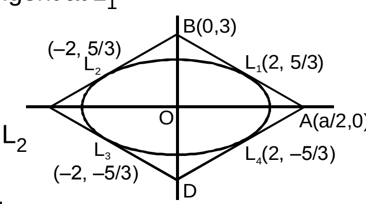
$e^2x = ae$

$x=\frac{a}{e}$ which the equation of directrix

Sol.7 (a) Ends of L.R. $\left(\pm ae,\pm\frac{b^2}{a}\right)$

$$\left(\pm 2,\pm\frac{5}{3}\right)$$

Equatoin of tangent at L_1



$$\frac{2x}{9}+\frac{y}{3}=1$$

equation of tangent at L_2

$$-\frac{2x}{9}+\frac{y}{3}=1$$

@ L_3

$$\frac{-2x}{9}-\frac{y}{3}=1$$

@ L_4

$$\frac{2x}{9}-\frac{y}{3}=1$$

Area of ΔOAB = $\frac{1}{2}\,\frac{9}{2}\,(3)=\frac{27}{4}$

Area of Quadrilateral = $4\times\frac{27}{4}$ = 27 Sq. units.

(b) Ellipse $\frac{x^2}{27}+y^2=1$ $a=3\sqrt{3}$ $b=1$

Tangent at point P (a cosθ, b sin θ)

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

Intercept on x-axis = $\frac{a}{\cos\theta}$

Intercept on y-axis = $\frac{b}{\sin\theta}$

$S = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$

$S = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$

$\frac{ds}{d\theta} = 0 \Rightarrow \theta = \frac{\pi}{6}$

S will be least at $\theta = \frac{\pi}{6}$

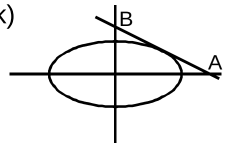
Sol.8 $\frac{x^2}{2} + \frac{y^2}{1} = 1$

Equation of tangent P(θ) $\qquad a = \sqrt{2}$

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad b = 1$

$A\left(\frac{a}{\cos\theta}, 0\right)$ $B\left(0, \frac{b}{\sin\theta}\right)$

Let the middle point M (h, k)



$2h = \frac{a}{\cos\theta} \Rightarrow \cos\theta = \frac{a}{2h}$

$2k = \frac{b}{\sin\theta} \Rightarrow \sin\theta = \frac{b}{2k}$

Square & add

$\frac{a^2}{4h^2} + \frac{b^2}{4k^2} = 1$

$\frac{2}{4h^2} + \frac{1}{4k^2} = 1$

$\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

Sol.9 **(a)** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Tangent at P(Q)

$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Meets the coordinates axes at

$A\left(\frac{a}{\cos\theta}, 0\right)$ $B\left(0\frac{b}{\sin\theta}\right)$

Area = $\frac{1}{2}\left(\frac{a}{\cos\theta}\right)\left(\frac{b}{\sin\theta}\right)$

$= \frac{ab}{\sin 2\theta} \geq ab$

(b) Tangent for circle

$y = mx + 4\sqrt{1+m^2}$ ...(1)

Tangent for ellipse

$y = mx + \sqrt{25m^2+4}$ ...(2)

(1) and (2) represent same line

$4\sqrt{1+m^2} = \sqrt{25m^2+4}$ ...(3)

$16 + 16m^2 = 25m^2 + 4$

$m = \pm\frac{2}{\sqrt{3}}$

Since Tangent is in 1st quad so m < 0

$m = -\frac{2}{\sqrt{3}}$ Equation of common tangent

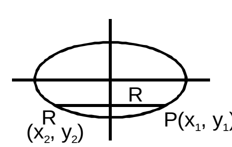
$y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$

Meets the coordinates axes at

$A(2\sqrt{7}, 0)$ & $\left(0, 4\sqrt{\frac{7}{3}}\right)$

$AB = \frac{14}{\sqrt{3}}$

Sol.10 $\frac{x^2}{4} + \frac{y^2}{1} = 1$



$b^2 = a^2(1 - e^2)$

$\Rightarrow \qquad e = \frac{\sqrt{3}}{2}$

$P\left(\sqrt{3}, \frac{-1}{2}\right)$ & $Q\left(-\sqrt{3}, \frac{-1}{2}\right)$

mid point of PQ

R $\left(0, \frac{-1}{4}\right)$

PQ = $2\sqrt{3}$ = length of LR

Two parabolas are possible whose vertices are

$\left(0, \frac{-\sqrt{3}}{2} - \frac{1}{2}\right)$ and $\left(0, \frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

Hence the equation of parabola are

$x^2 - 2\sqrt{3}\, y = 3 + \sqrt{3}$

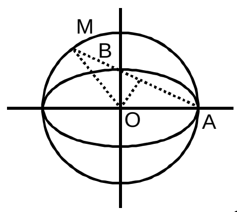
$x^2 + 2\sqrt{3}\, y = 3 - \sqrt{3}$

Sol.11 $\frac{x^2}{9} + y^2 = 1$

A(3, 0), B(0, 1)

Equation of line AM is

$x + 3y - 3 = 0$



Perpendicular distance of line from origin = $\frac{3}{\sqrt{10}}$

length of AM = $2\sqrt{9 - \frac{9}{10}} = \frac{18}{\sqrt{10}}$

Area = $\frac{1}{2} \times \frac{18}{\sqrt{10}} \times \frac{3}{\sqrt{10}} = \frac{27}{10}$ Sq. units.

Sol.12 Normal is $4x \sec\phi - 2y \operatorname{cosec}\phi = 12$

Q $(3\cos\phi, 0)$

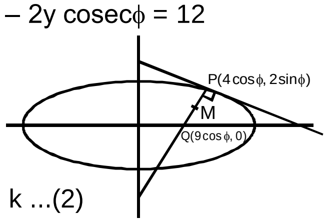
M : (h, k)

$2h = 3\cos\phi + 4\cos\phi$

$2h = 7\cos\phi$ ...(1)

$2k = 2\sin\phi \Rightarrow \sin\phi = k$ ...(2)

square & add



$\frac{4h^2}{49} + k^2 = 1 \Rightarrow \frac{4x^2}{49} + y^2 = 1$

Latus rectum $x = \pm 2\sqrt{3}$

$\frac{48}{49} + y^2 = 1 \Rightarrow y = \pm \frac{1}{7}$

points $\left(\pm 2\sqrt{3}, \pm \frac{1}{7}\right)$

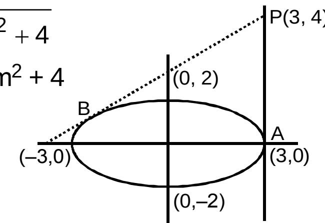
Sol.13 D

$y = mx + \sqrt{9m^2 + 4}$

$4 - 3m = \sqrt{4m^2 + 4}$

$16 + 9m^2 - 24m = 9m^2 + 4$

$m = \frac{1}{2}$



Equaton is $y - 4 = \frac{1}{2}(x - 3)$

$2y - 8 = x - 3 \Rightarrow x - 2y + 5 = 0$ ...(1)

Let B = $(\alpha, \beta) \Rightarrow \frac{x\alpha}{9} + \frac{y\beta}{4} - 1 = 0$ ...(2)

(1) & (2) are same

$\frac{\frac{\alpha}{9}}{1} = \frac{\frac{\beta}{4}}{-2} = \frac{-1}{5}$

$\Rightarrow \alpha = -\frac{9}{5}, \beta = \frac{8}{5}$ so B $\left(\frac{-9}{5}, \frac{8}{5}\right)$

Sol.14 C

Let orthocentre D

slope of D must be zero

$\Rightarrow \frac{y-8}{5} = 0\left(x + \frac{9}{5}\right) \Rightarrow y = \frac{8}{5}$

Hence y-coordinate of D is $\frac{8}{5}$

Sol.15 A

Locus will be parabola

Equation of AB is $\frac{3x}{9} + \frac{4y}{4} = 1$

$\Rightarrow \frac{x}{3} + y = 1 \Rightarrow x + 3y - 3 = 0$

$(x-3)^2 + (y-4)^2 = \frac{(x + 3y - 3)^2}{10}$

$10x^2 + 90 - 60x + 10y^2 + 160 - 80y = x^2 + 9y^2 + 9 + 6xy - 6x - 18y$

$\Rightarrow 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$